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Name of College - S.S. College
Jehanabad

Deptt - Mathematics

Topics - polar Asymptotes

Class - B.Sc (Hons)

Time - 10:15 AM to 11 AM Date - 19-08-2020
11.00 AM to 11.45 AM 20-08-2020

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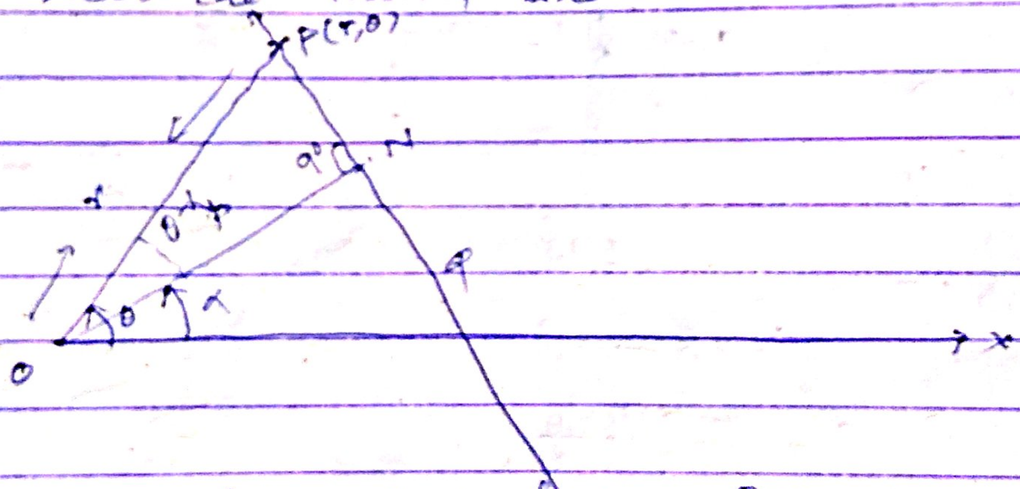
Asymptotes of polar Curve :→

(1) To Find the polar Equation of a St. line

$$p = r\cos(\theta - \alpha)$$

Where p = length of the perpendicular from the pole to the St. line and α = angle which this St. line makes with the initial line



Let PQ be any line. From From O Draw ON perpendicular to PQ where ON = p

Say ON makes an angle α with the +ve direction with OX

Let $P(r, \theta)$ be any point.
clearly $OP = r$
$\angle POX = \theta$

$\angle NOP = \theta - \alpha$

Now from right angled triangle OPN

$$\cos(\theta - \alpha) = \frac{ON}{OP}$$

$$\Rightarrow \cos(\theta - \alpha) = \frac{p}{r}$$

$$\Rightarrow p = r\cos(\theta - \alpha)$$

this is polar equation of the st. line

Asymptotes of the polar curve

Equation of curve $r = f(\theta)$

Equation of st. line
$$p = r\cos(\theta - \alpha)$$

Method of finding polar asymptote —

(I) put $r = \frac{1}{u}$ in the given equation and find the limit of θ when $u \to 0$

(II) Let $\theta = \theta_1$ be any one limit among the various possible limits of θ

(III) Find limit of $\left(-\frac{d\theta}{du}\right)$ when $u \to 0$, $\theta \to \theta_1$

Let this limit be p

then $p = r.\sin(\theta_1 - \theta)$ will be the equation of asymptote

Find the asymptotes of

$r = a/\theta$ —— (1)

When $r = \infty$

(1) $\Rightarrow \theta = 0 = \alpha$

$$\frac{dr}{d\theta} = -\frac{a}{\theta^2}$$

$$\therefore \frac{d\theta}{dr} = -\frac{\theta^2}{a}$$

$$\Rightarrow r^2\frac{d\theta}{dr} = -\frac{r^2\theta^2}{a}$$

$$\lim_{\theta \to 0} r^2\frac{d\theta}{dr} = -\lim_{\theta \to 0} r^2\frac{\theta^2}{a}$$

$$\therefore r = \frac{a}{\theta}$$

$$r^2 = \frac{a^2}{\theta^2}$$

$$= -\lim_{\theta \to 0} \frac{a^2}{\theta^2}\cdot\frac{\theta^2}{a}$$

$$= -a$$

$$\therefore p = \lim_{\theta \to \alpha} r^2\frac{d\theta}{dr} = -a$$

Now using formula

$$p = r\sin(\alpha - \theta)$$

$$-a = r\sin(0 - \theta)$$

$$= -r\sin\theta$$

$a = r\sin\theta$ Which is required asymptote

(i) Find the Asymptotes

when $r = \frac{a}{\sqrt{\theta}}$

Solution

Here $r = \frac{a}{\sqrt{\theta}}$

when $r = \infty$, $\theta = 0$ i.e. $\alpha = 0$.

$$\frac{dr}{d\theta} = a\left(-\frac{1}{2}\theta^{-3/2}\right)$$

$$= -\frac{1}{2}\frac{a}{\theta^{3/2}}$$

$$\therefore \frac{d\theta}{dr} = -\frac{2\theta^{3/2}}{a}$$

$$r^2\frac{d\theta}{dr} = -\frac{2r^2\theta^{3/2}}{a}$$

$$\because r = \frac{a}{\sqrt{\theta}}$$

$$r^2 = \frac{a^2}{\theta}$$

$$= -2\cdot\frac{a^2}{\theta}\cdot\frac{\theta^{3/2}}{a}$$

$$r^2\frac{d\theta}{dr} = -2a\theta^{1/2}$$

$$\therefore p = \lim_{\theta\to\alpha} r^2\frac{d\theta}{dr}$$

$$= \lim_{\theta\to 0} -2a\theta^{1/2}$$

$$= 0$$

$p = 0$

Now using formula

$$p = r\sin(\alpha - \theta)$$

$$0 = r\sin(-\theta) \Rightarrow \boxed{\sin\theta = 0} \Rightarrow \theta = 0$$

Which is required Asym

Find the asymptotes of

$r = a\,\mathrm{cosec}\,\theta + b$. — i

Here $r = a\,\mathrm{cosec}\,\theta + b$ ——— (i)

$\Rightarrow r = \dfrac{a}{\sin\theta} + b$

$= \dfrac{a + b\sin\theta}{\sin\theta}$ ——— (ii)

put $r = \infty$

$\Rightarrow \sin\theta = 0 \Rightarrow \alpha = 0 \quad \beta = \pi$ (Taking principal values)

$\theta = 0, \pi$.

Now

$$\frac{dr}{d\theta} = -a\,\mathrm{cosec}\,\theta\,\cot\theta.$$

$$\frac{d\theta}{dr} = -\frac{1}{a\,\mathrm{cosec}\,\theta\,\cot\theta}.$$

$$r^2\frac{d\theta}{dr} = -r^2 \cdot \frac{1}{a\,\mathrm{cosec}\,\theta\,\cot\theta}$$

$$= -\frac{(a+b\sin\theta)^2}{\sin^2\theta} \cdot \frac{1}{a\,\mathrm{cosec}\,\theta\,\cot\theta}$$

$$= -\frac{(a+b\sin\theta)^2}{\cancel{\sin^2\theta}} \cdot \frac{\cancel{\sin^2\theta}}{a\cos\theta}$$

$$= -\frac{(a+b\sin\theta)^2}{a\cos\theta}$$

Value of p

When $\alpha = 0$

$$\lim_{\alpha\to 0} r^2\frac{d\theta}{dr} = -\lim_{\alpha\to 0}\frac{(a+b\sin\theta)^2}{a\cos\theta}$$

$p = -a \qquad = -\dfrac{a^2}{a} = -a$

By using formula

$$p = r\sin(\alpha - \theta)$$

$$-a = r\sin(0 - \theta)$$

$$= -r\sin\theta$$

$$a = r\sin\theta$$

When $\beta = \pi$

$$\lim_{\theta \to \pi} r^2 \frac{d\theta}{dr} = \lim_{\theta \to \pi} \frac{(a + b\sin\theta)^2}{a\cos\theta}$$

$$= \frac{a^2}{a} = a$$

Now using

$$p = r\sin(\alpha - \theta)$$

$$+a = r\sin(\pi - \theta)$$

$$= r\sin\theta$$

$a = r\sin\theta$ which is required asymptotes

Q. Find the asymptotes when the Equations of curve

$$r = 2a\sin\theta\tan\theta$$

Solution - Equations of curve

$$r = 2a\sin\theta\tan\theta$$

$$= \frac{2a\sin^2\theta}{\cos\theta}$$

When $r = \infty$ $\quad \cos\theta = 0$

$$\theta = \pi/2,\ 3\pi/2$$

i.e $\alpha = \pi/2,\ \beta = \frac{3\pi}{2}$

$\frac{\sin\theta}{\cos\theta}$

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Diff. (1) with respect to θ

$$\frac{dr}{d\theta} = 2a\left[\frac{\cos\theta \times 2\sin\theta\cos\theta - \sin^2\theta(-\sin\theta)}{\cos^2\theta}\right]$$

$$= 2a\left[\frac{2\sin\theta\cos^2\theta + \sin^3\theta}{\cos^2\theta}\right]$$

$$\therefore \frac{d\theta}{dr} = \frac{\cos^2\theta}{2a[2\sin\theta\cos^2\theta + \sin^3\theta]}$$

$$r^2\frac{d\theta}{dr} = \frac{\cancel{4}^{2}a^{\cancel{2}} \sin^4\theta}{\cancel{\cos^2\theta}} \times \frac{\cancel{\cos^2\theta}}{\cancel{2a}[2\sin\theta\cos^2\theta + \sin^3\theta]}$$

$$= \frac{2a\sin^3\theta}{2\cos^2\theta + \sin^2\theta}$$

When $\alpha = \pi/2$ $= \frac{2a\sin^3\theta}{2\cos^2\theta + \sin^2\theta}$

$$\therefore p = \lim_{\theta\to\pi/2} \frac{2a\sin^3\theta}{2\cos^2\theta + \sin^2\theta}$$

$$p = \lim_{\theta\to\pi/2} r^2\frac{d\theta}{dr} = 2a$$

Now using formula

$$p = r\sin(\alpha - \theta)$$

$$2a = r\sin(\pi/2 - \theta)$$
$$= r\cos\theta$$

Which is the required asymptote

Again

$$p = \lim_{\theta \to 3\pi/2} \frac{2a\sin^3\theta}{2\cos^2\theta + \sin^2\theta}$$

$$= \frac{2a(-1)^3}{0 + (-1)^2}$$

$$= -2a \qquad \text{When } \beta = 3\pi/2$$

Now applying the formula

$$p = r\sin(\beta - \theta)$$

$$= r\sin\left[\frac{3\pi}{2} - \theta\right]$$

$$-2a = -r\cos\theta$$

$$\Rightarrow 2a = r\cos\theta$$

Hence Required asymptotes

$$2a = r\cos\theta$$

Find the asymptotes of the curve

$$r\sin 2\theta = a\cos 3\theta$$

Solution: Equation of curve

$$r\sin 2\theta = a\cos 3\theta$$

$$r = \frac{a\cos 3\theta}{\sin 2\theta} \quad \text{—— (1)}$$

put $r = \infty$ in (1)

$$\Rightarrow \sin 2\theta = 0$$

$$2\theta = \sin^{-1} 0 = 0$$

$$2\theta = 0, \qquad \theta = 0$$

$$2\theta = \pi \qquad \theta = \pi/2$$

Case $\alpha = 0$, $\cancel{\beta} = \frac{\pi}{2}$

Now

$$r = \frac{a\cos 3\theta}{\sin 2\theta}$$

Diff. with respect to θ.

$$\frac{dr}{d\theta} = a\left[\frac{\sin 2\theta(-3\sin 3\theta) - \cos 3\theta \times 2\cos 2\theta}{\sin^2 2\theta}\right]$$

$$= -a\left[\frac{3\sin 3\theta \sin 2\theta + 2\cos 2\theta \cos 3\theta}{\sin^2 2\theta}\right]$$

Now $\frac{d\theta}{dr} = -\frac{\sin^2 2\theta}{a[3\sin 3\theta \sin 2\theta + 2\cos 2\theta \cos 3\theta]}$

$$r^2\frac{d\theta}{dr} = -\frac{a^2\cos^2 3\theta}{\cancel{\sin^2 2\theta}} \cdot \frac{\cancel{\sin^2 2\theta}}{a(3\sin 3\theta \sin 2\theta + 2\cos 2\theta \cos 3\theta)}$$

$$= -\frac{a\cos^3\theta}{3\sin 3\theta \sin 2\theta + 2\cos 2\theta \cos 3\theta}$$

When $\alpha = 0$

$$p = \lim_{\theta \to 0} r^2 \frac{d\theta}{dr}$$

$$= \lim_{\theta \to 0} -\frac{a\cos^3\theta}{3\sin 3\theta \sin 2\theta + 2\cos 2\theta \cos 3\theta}$$

$$= -\frac{a}{2}$$

Now using Formula

$p = r\sin(\alpha - \theta)$

$\Rightarrow -\frac{a}{2} = r\sin(0-\theta)$

$= -r\sin\theta$

$\Rightarrow a = 2r\sin\theta.$ is the required Asymptotes

Value of p when $\beta = \pi/2$

$p = \lim_{\theta \to \pi/2} r^2 \frac{d\theta}{dr}$

$= \lim_{\theta \to \pi/2} - \frac{a\cos^3\theta}{3a\sin 3\theta \sec^2\theta + 2\cos 2\theta \cos 3\theta}$

$= 0$

Now Asymptotes when $\beta = \pi/2$

$p = r\sin(\beta - \theta)$

$0 = r\sin(\pi/2 - \theta)$

$= r\cos\theta$

$\Rightarrow \cos\theta = 0$ or $\theta = \cos^{-1}(0)$

$= \pi/2$,

$\theta = \pi/2$